An Aspect of Oscillatory Conditions in Linear Systems and Hopf Bifurcations in Nonlinear Systems

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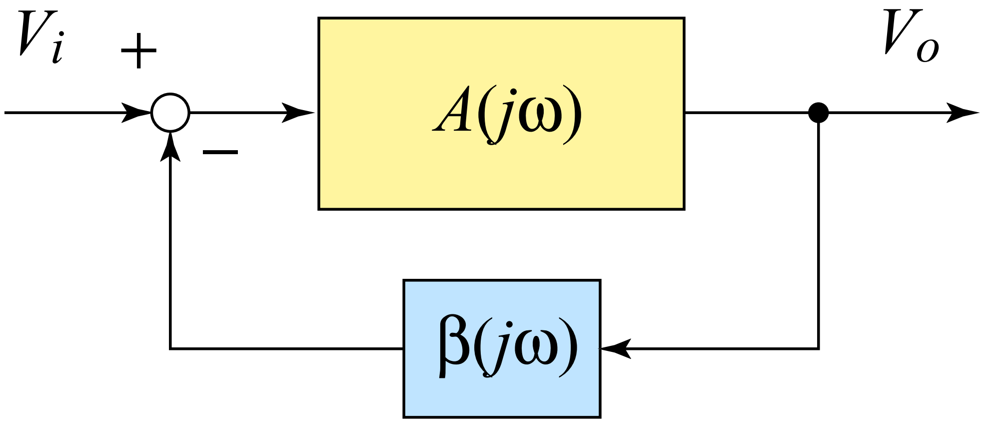
Background

Oscillators — limit cycles in nonlinear circuits

- ✎ **generation: lost stability for equilibrium — Hopf bifurcation**
- ✎ **required structural stability**
- ✎ **design problem**
  - ✏ **realization: RC phase shift or LC loop**
  - ✏ **design parameters: physical elements, wave shape and frequency**

Oscillator in text books



1. **provide an amplifier** $A(j\omega)$ **and a positive feedback** $\beta(j\omega)$**.**
2. **the trigger can be** *noise* **in the circuitry.**
3. **the output is applied into the input** *in phase.*
4. **no external force is needed anymore.**

Transfer function expression

The transfer function of the feedback system:

$$H(j\omega) = \frac{A(j\omega)}{1 - A(j\omega)\beta(j\omega)}$$

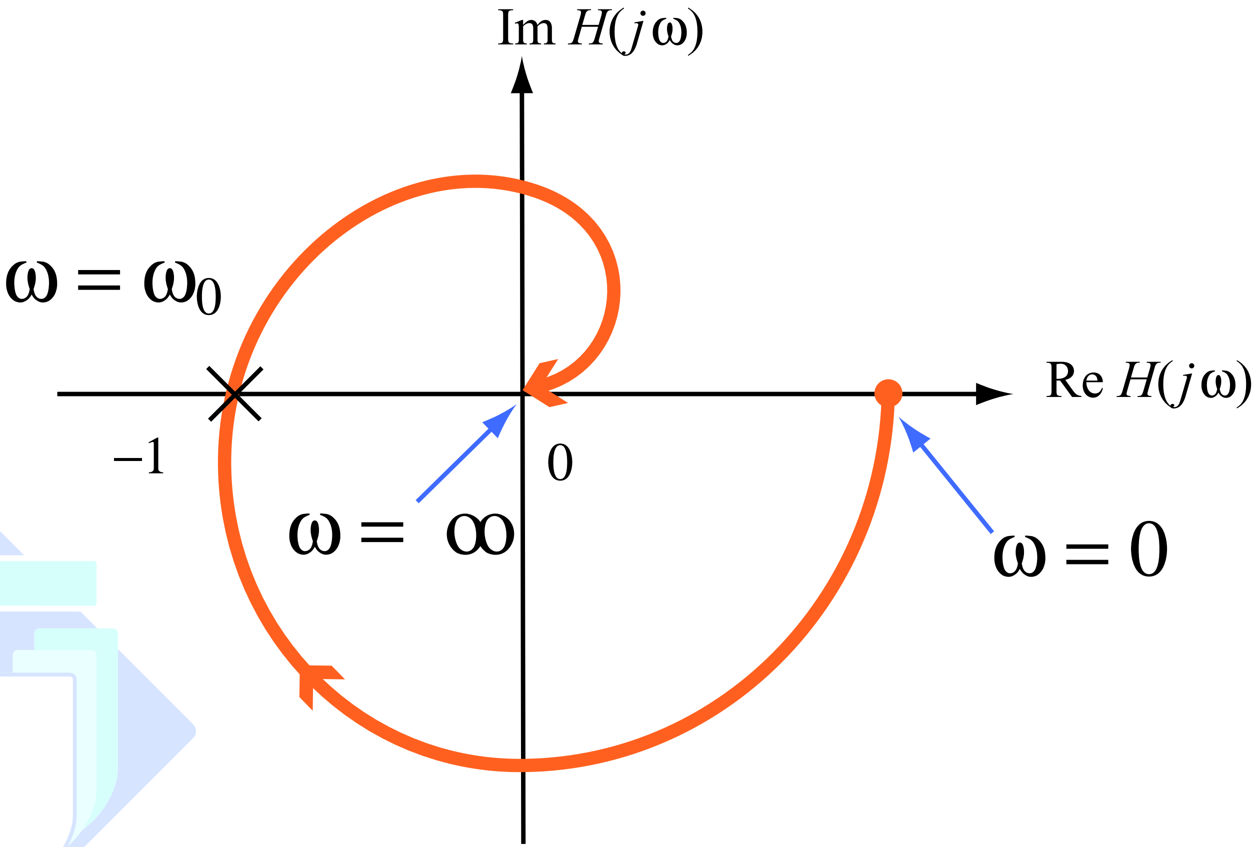
If $A\beta \approx 1, \exists\omega = \omega_0$**, then** $H(j\omega_0) = \infty$**.**

Barkhausen criterion

$$\mathbf{Re} A(j\omega_0)\beta(j\omega_0) = 1, \quad \mathbf{Im} A(j\omega_0)\beta(j\omega_0) = 0$$

Polar plots

Barkhausen criterion of Nyquist plot:



$$\mathbf{Re}\,A(j\omega_0)\beta(j\omega_0) = 1, \quad \mathbf{Im}\,A(j\omega_0)\beta(j\omega_0) = 0$$

Linear and nonlinear systems

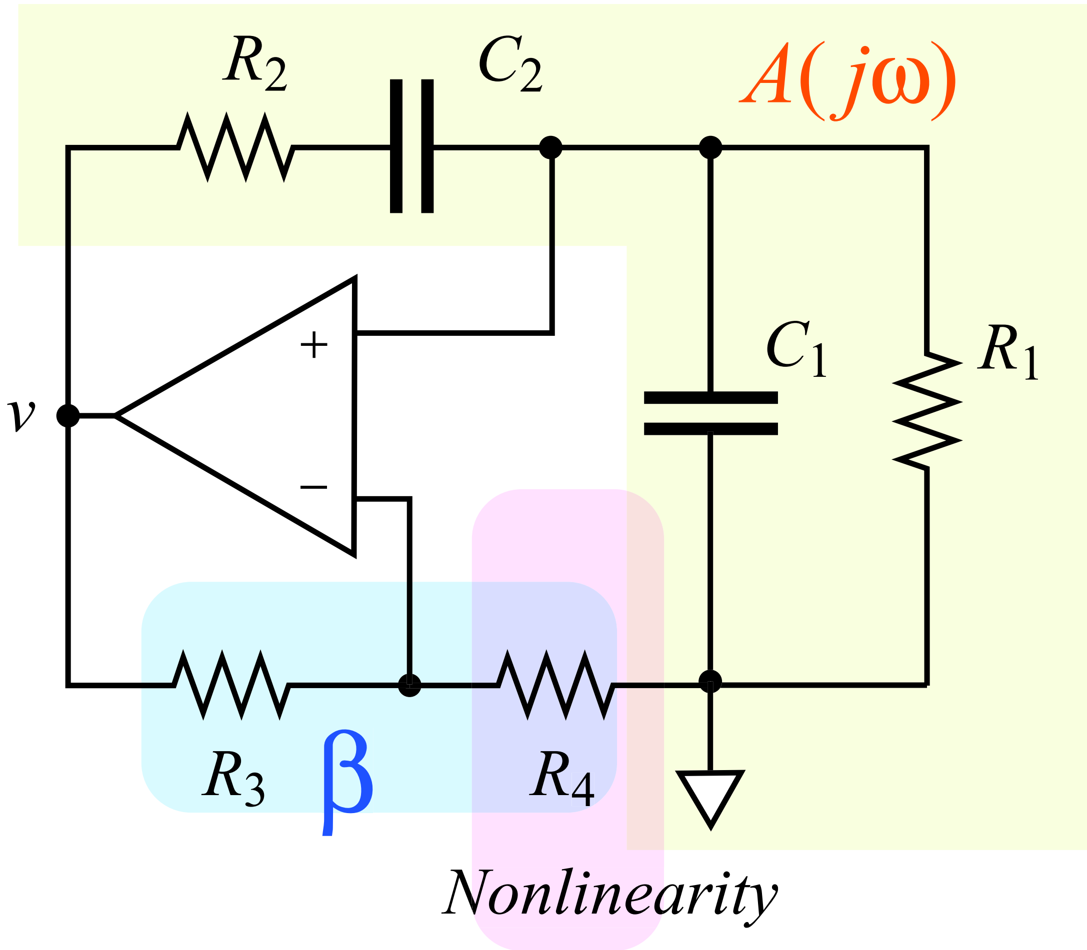
Barkhausen criterion for linear systems gives a harmonic oscillation — structurally unstable !

$\Downarrow$

Need a gimmick to suppress amplitude growing.

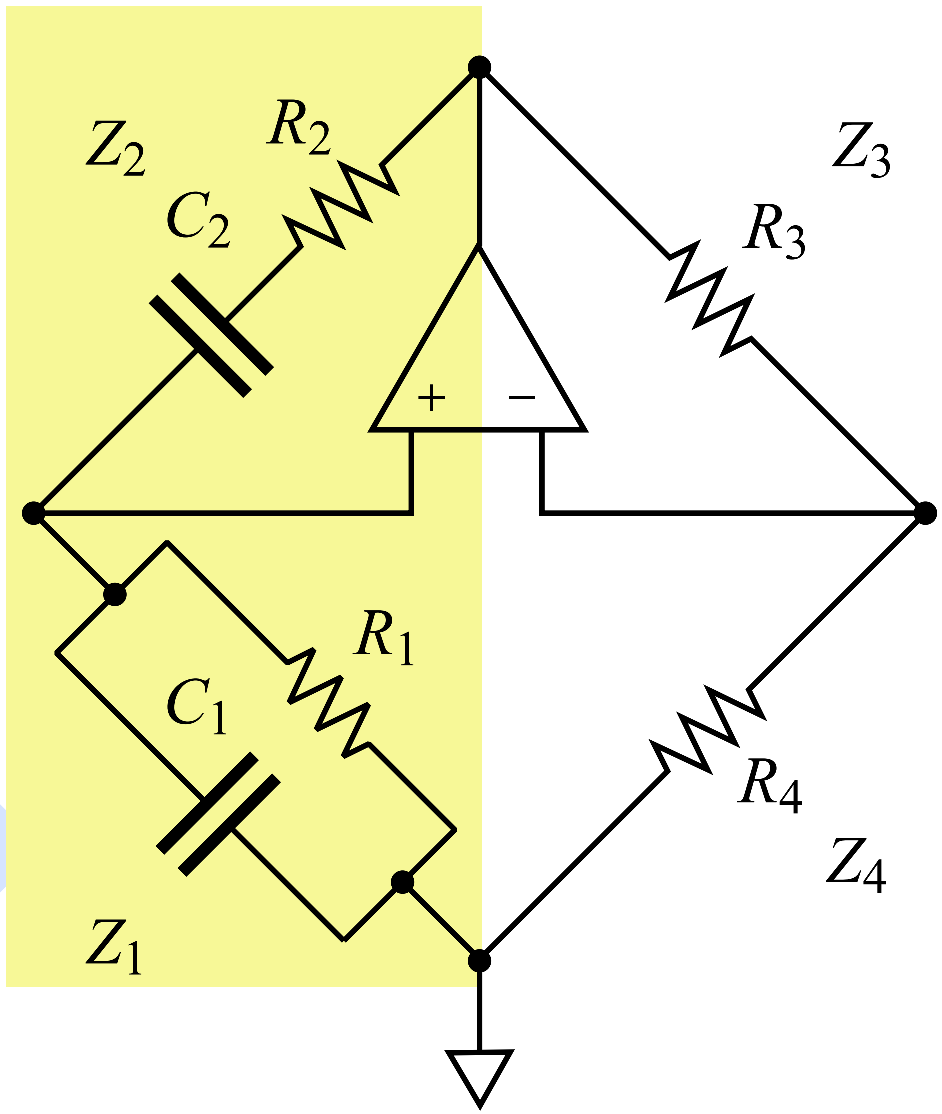
- ✎ **To retain the stability in large: nonlinearity— a natural assumption: limiter, clamper, thermistor**
- ✎ **getting the structural stability**

Wien bridge oscillator



R_4 should be nonlinear to keep stable oscillation.

Wien bridge oscillator



$$Z_1 Z_3 = Z_2 Z_4$$

In this talk . . .

Design an oscillator — derive a condition for a given linear circuit.
We show that the following factors are equivalent:

- ✎ **creation of zero impedance**
- ✎ **a current flows without power source — virtual source method**
- ✎ **Barkhausen criterion**
- ✎ **Hopf bifurcation**

Virtual source method

Assumption of a virtual source by using the phasor method:

- ✎ **Adding a virtual voltage (current) source into the system**
- ✎ **zero impedance (admittance) makes a non-zero current (voltage) without source**

Virtual voltage source

- ✎ place a **virtual voltage source** $E = E_0 e^{j\omega t}$ **in an appropriate location in the circuit**
- ✎ **compute the whole impedance** Z
- ✎ $ZI = E$ **shows "a condition to keep the current non-zero is** $Z = 0$**."**

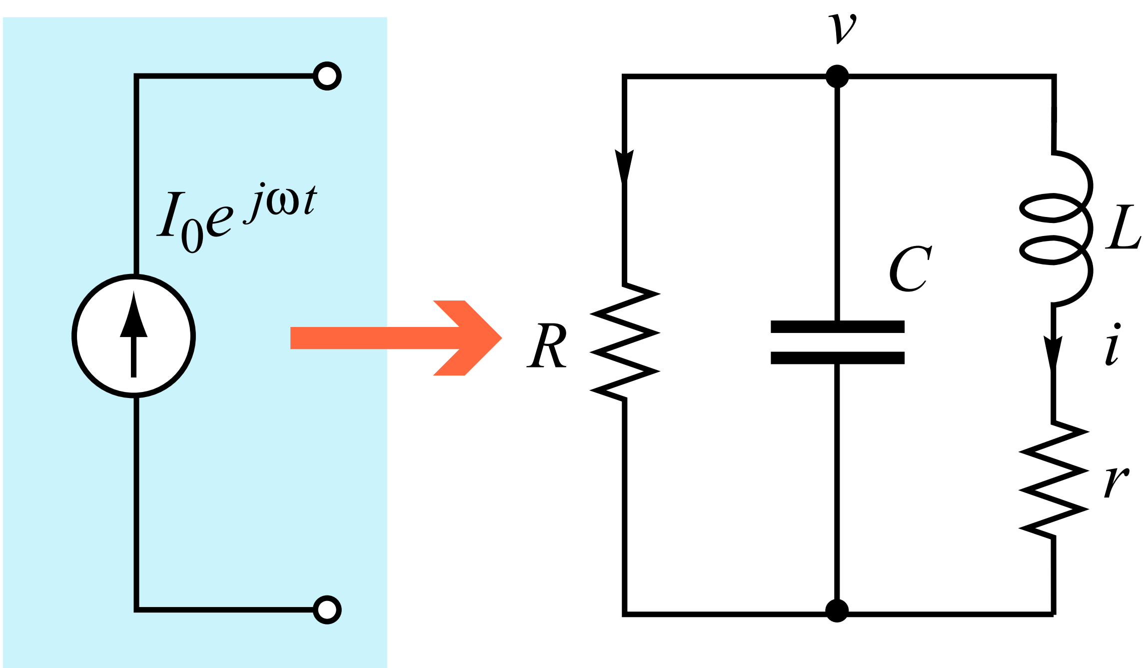
Virtual current source

- ✎ place a virtual current source $I = I_0 e^{j\omega t}$ in an appropriate location in the circuit
- ✎ compute the whole admittance Y
- ✎ $YV = I$ shows "a condition to keep the voltage non-zero is $Y = 0$."

These virtual source methods are equivalent to Hopf bifurcation analysis and Barkhausen criterion.

Example—LCR circuit

adding a virtual current source



$$Z_1 = (r_1 + j\omega L_1) \parallel \frac{1}{j\omega C} \parallel R$$

$$= \frac{R(r_1 + j\omega L_1)}{r_1 + R - \omega^2 R L_1 C + j\omega(L_1 + r_1 RC)}$$

LCR circuit

substitute $0 + j0$ **into the denominator of** Z_1

$$R + r_1 - \omega^2 R L_1 C = 0$$

$$\omega(L_1 + R r_1 C) = 0$$

since $L_1 > 0$**,** $C > 0$**, we have**

$$\omega = \sqrt{\frac{R + r_1}{R L_1 C}}$$ **frequency**

$$R r_1 = -\frac{L_1}{C}$$ **Hopf bifurcation set**

$R < 0$ **required.**

Hopf bifurcation analysis

Assume $R^{-1}v = g(v)$ **Hopf bifurcation analysis**

$$C\frac{dv}{dt} = -g(v) - i, \quad L_1\frac{di}{dt} = v - ri$$

Jacobian matrix:

$$J = \begin{pmatrix} -\dfrac{1}{CR} & -\dfrac{1}{C} \\ \dfrac{1}{L_1} & -\dfrac{r}{L_1} \end{pmatrix}.$$

where R^{-1} **is a linear part of** $dg(v)/dt$

$$\chi(\mu) = \det(A - \mu I) = 0$$

$$\chi(\mu) = \mu^2 + \left(\frac{1}{CR} + rL_1\right)\mu + \frac{R + r}{CRL_1} = 0$$

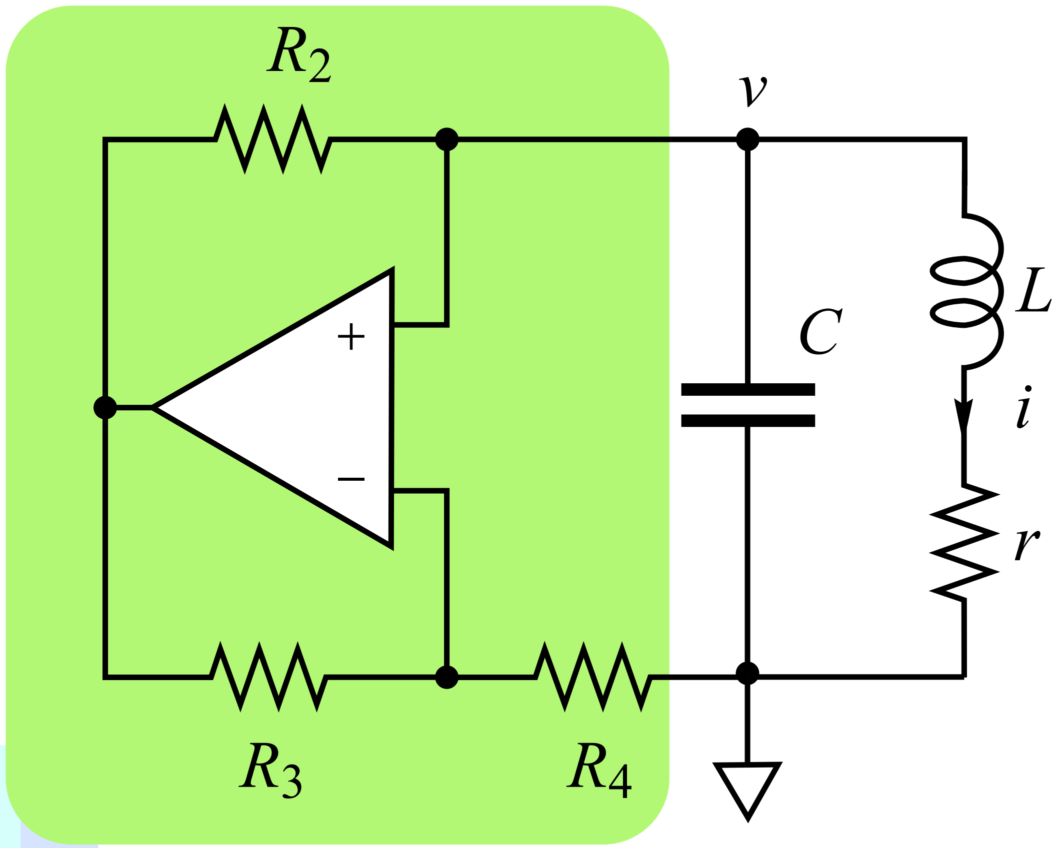
Substitute Hopf bifurcation condition: $\mu = 0 + j\omega$ **into above equation.**

From the real and imaginary part, we have:

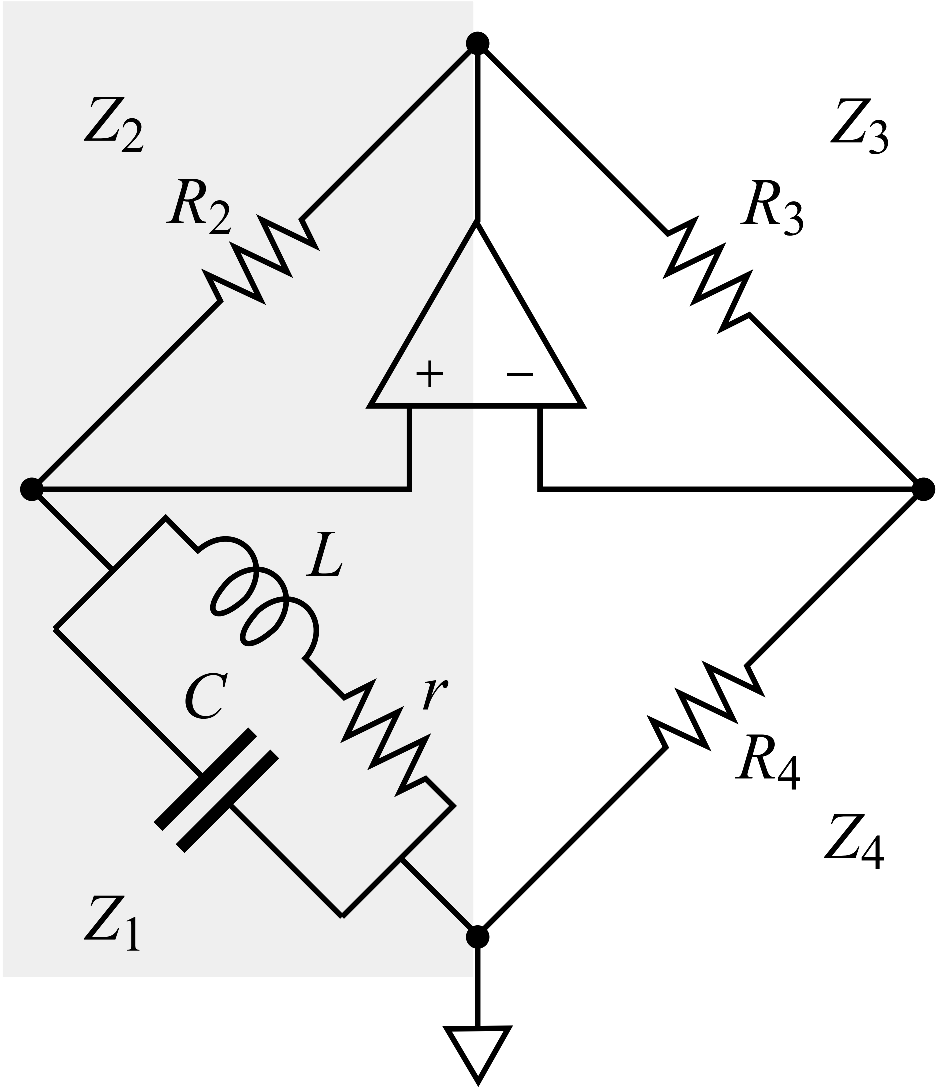
$$\omega = \sqrt{\frac{R + r_1}{RL_1C}}$$

$$Rr_1 = -\frac{L_1}{C}$$

Barkhausen criterion for BVP



as a bridge



Voltage feedback ratio

for the positive feedback loop:

$$\beta = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2(1 - \omega^2 L_1 C + j\omega C r)}{r + R_2 - \omega^2 L_1 C R_2 + j\omega L_1 + j\omega C r R_2}$$

Let $R = R_3 = R_4$**, and from Re**$A\beta = 1$**, Im**$A\beta = 0$**, we have:**

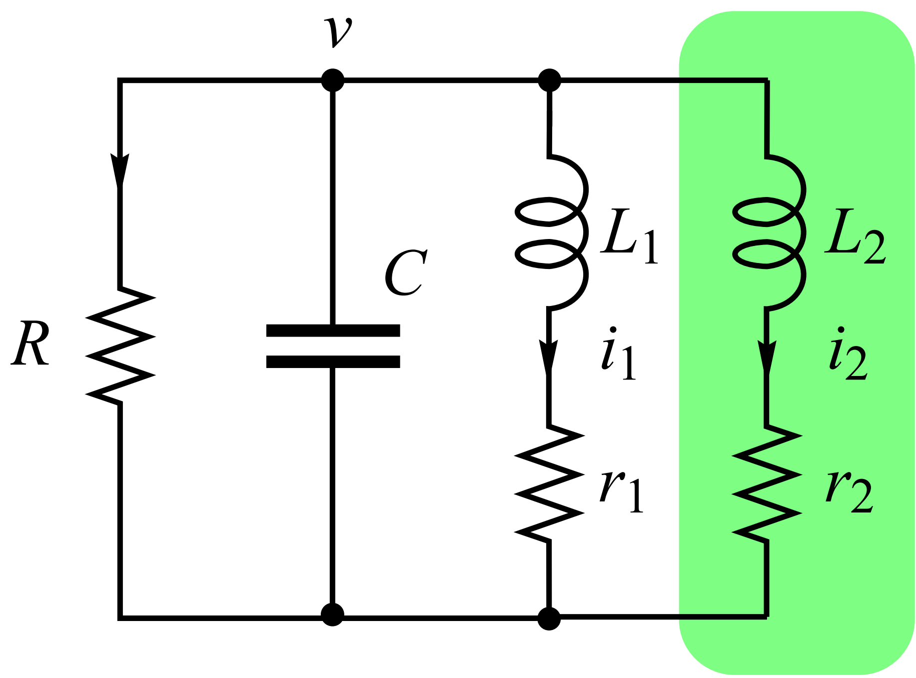
$$\omega = \sqrt{\frac{R + r_1}{R L_1 C}}, \quad R r_1 = -\frac{L_1}{C}$$

⇒ showing the same result.

Cautions

- ✎ **Controlling Hopf bifurcation by choosing parameter values**
- ✎ **No guarantee for global stability of the limit cycle**
- ✎ **post-process: getting global stability, e.g., replacing *R* by a nonlinear conductor**
- ✎ **Frequency and wave shape — simulation is needed**

Application — extension of BVP



virtual source method:

$Z_2 = Z_1 \parallel (r_2 + j\omega L_2)$

$$R(r_1 + r_2) + r_1 r_2 - \omega^2 (r_2 R L_1 C + L_2 (r_1 R C + L_1)) = 0$$

$$R(L_1 + L_2) + r_1 L_2 + r_2 L_1 + r_1 r_2 R C - \omega^2 R L_1 L_2 C = 0$$

We have Hopf bifurcation curve $H(r_1, r_2)$, and

$$\omega = \sqrt{\frac{(r_1 + r_2)R + r_1 r_2}{L_1 L_2 + r_2 R C L_1 + r_1 R L_2 C}}$$

$r_2 \to \infty$

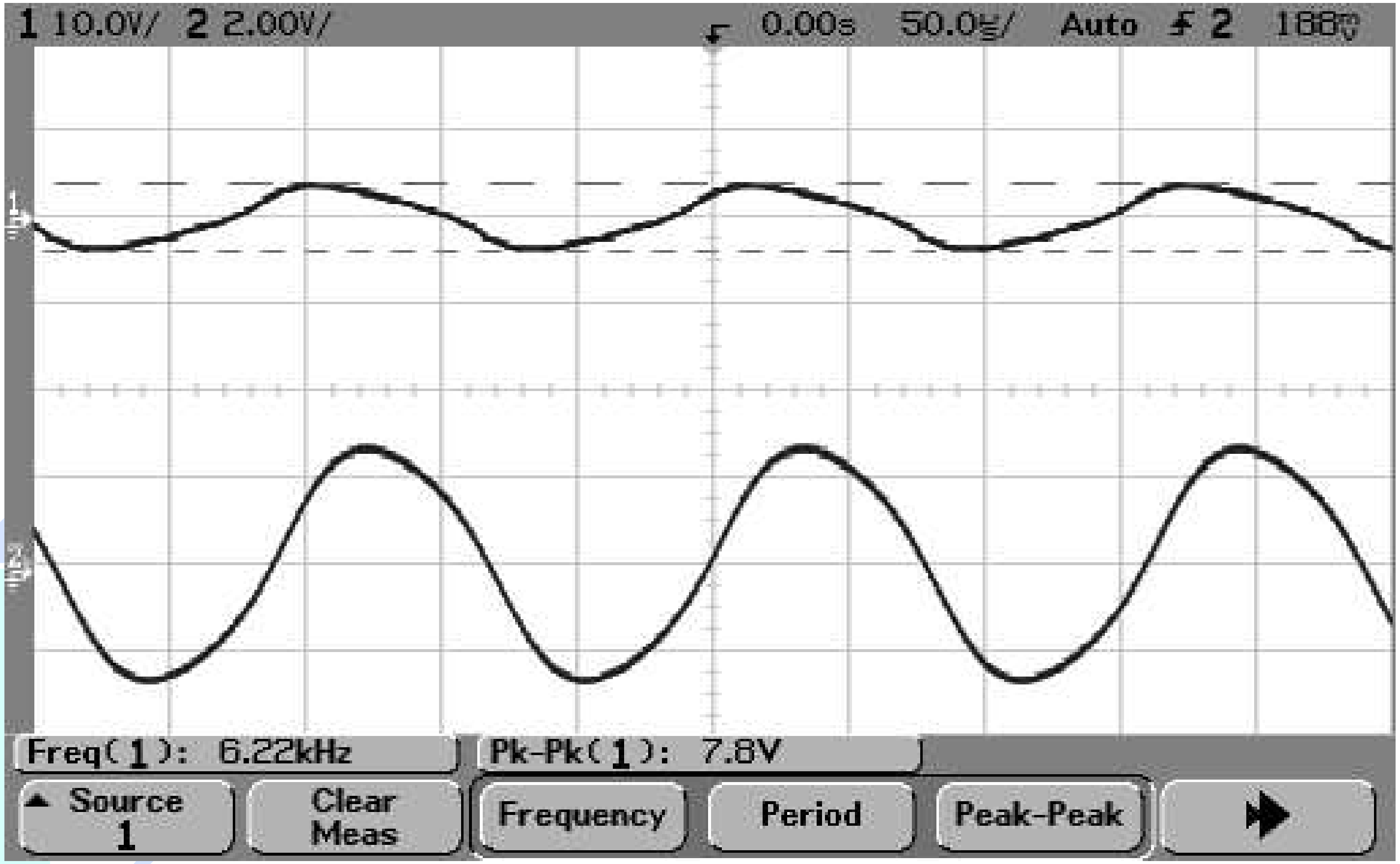
$$\omega_\infty = \sqrt{\frac{R + r_1}{R L_1 C}}$$

$r_2 \to 0$

$$\omega_0 = \sqrt{\frac{R r_1}{L_1 L_2 + R r_1 L_2 C}}$$

Slow oscillation (6 kHz)

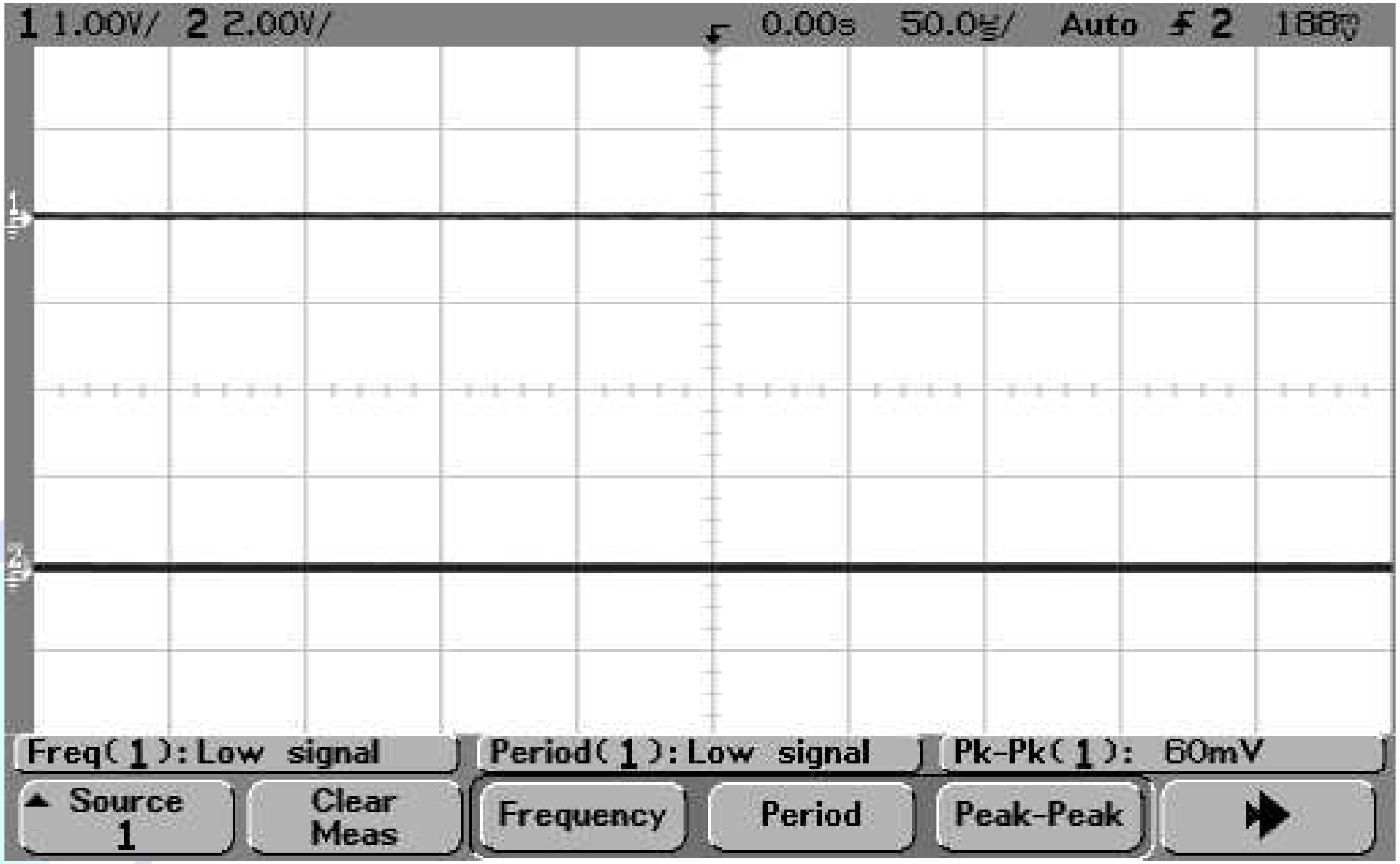
$$C = 0.022[\mu\mathbf{F}], r_1 = 500[\Omega], L_1 = 10[\mathbf{mH}], L_2 = 1[\mathbf{mH}]$$



(a) 100 kΩ

oscillation dead

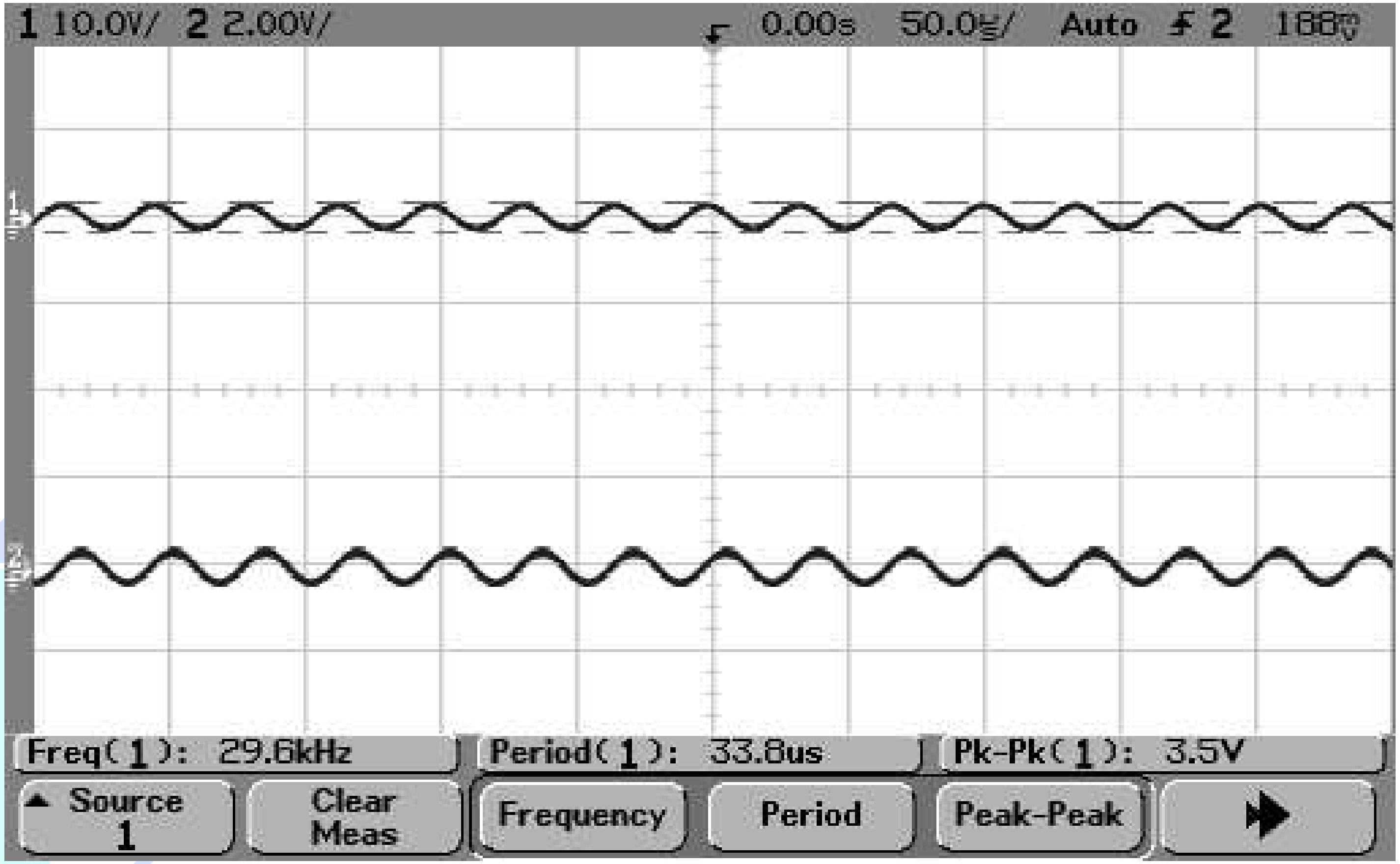
$$C = 0.022[\mu\mathbf{F}], r_1 = 500[\Omega], L_1 = 10[\mathbf{mH}], L_2 = 1[\mathbf{mH}]$$



(a) 400 Ω

revival

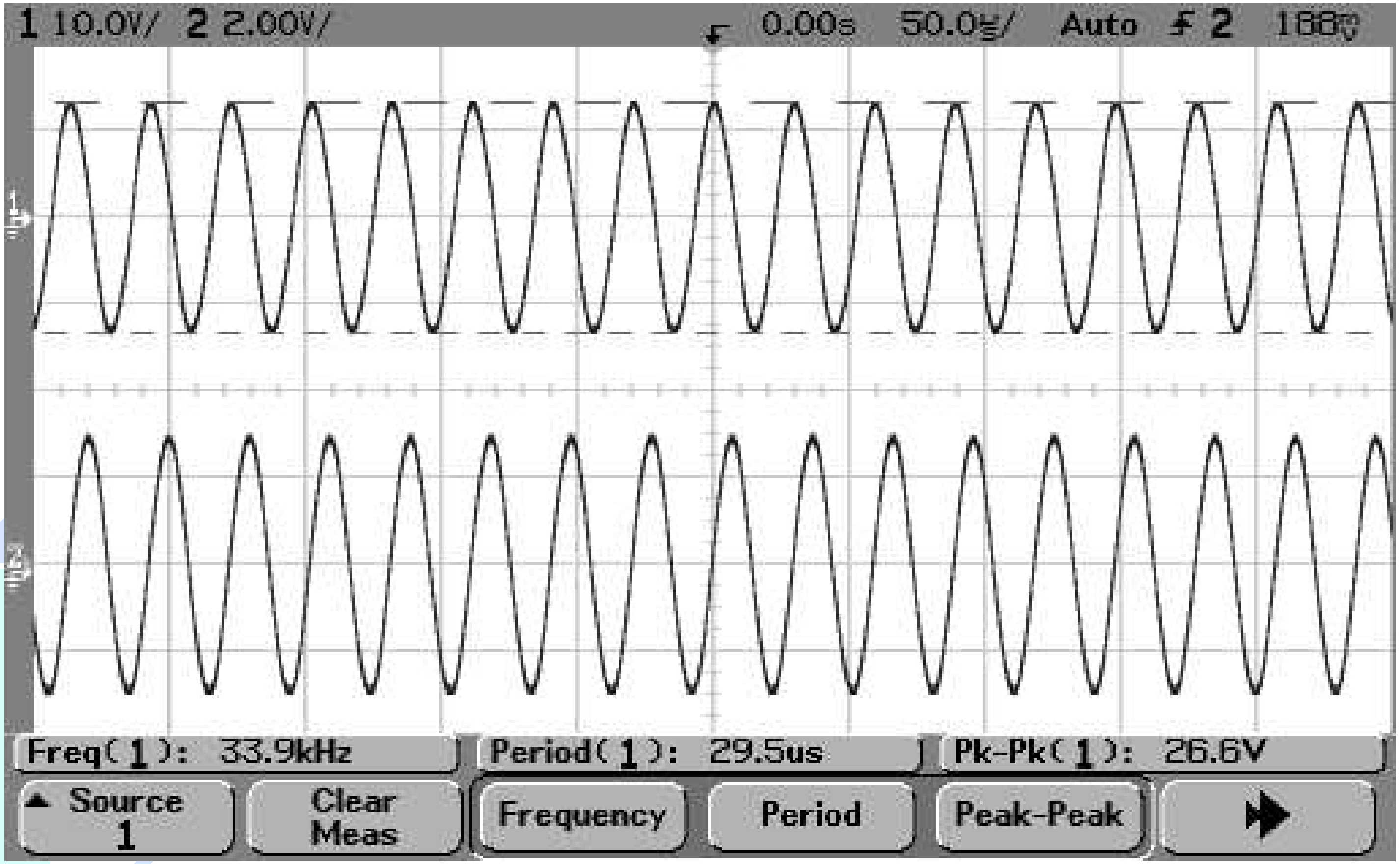
$$C = 0.022[\mu\mathbf{F}], r_1 = 500[\Omega], L_1 = 10[\mathbf{mH}], L_2 = 1[\mathbf{mH}]$$



(a) 100 Ω

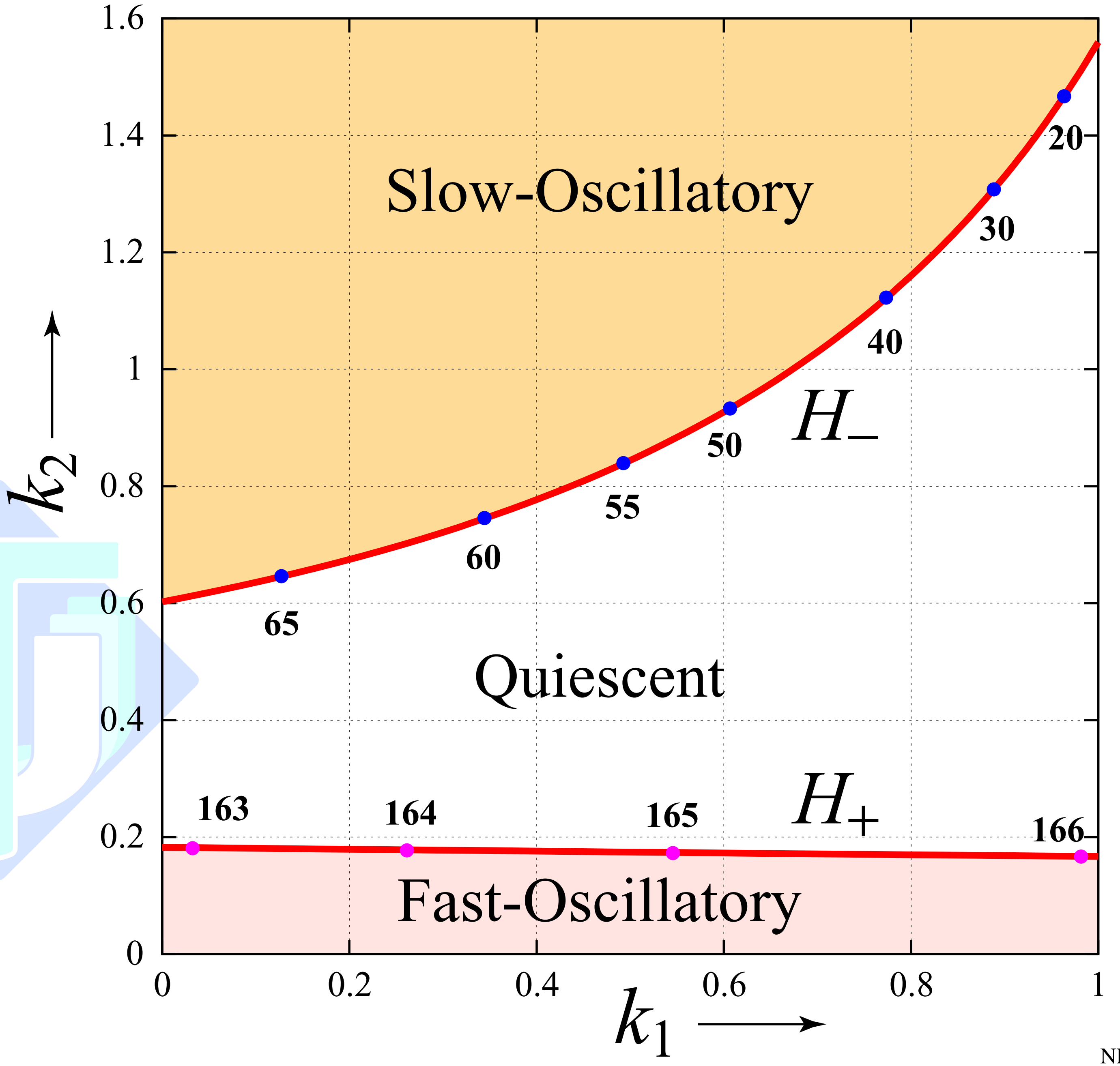
Fast oscillation (34 kHz)

$$C = 0.022[\mu\mathbf{F}], r_1 = 500[\Omega], L_1 = 10[\mathbf{mH}], L_2 = 1[\mathbf{mH}]$$

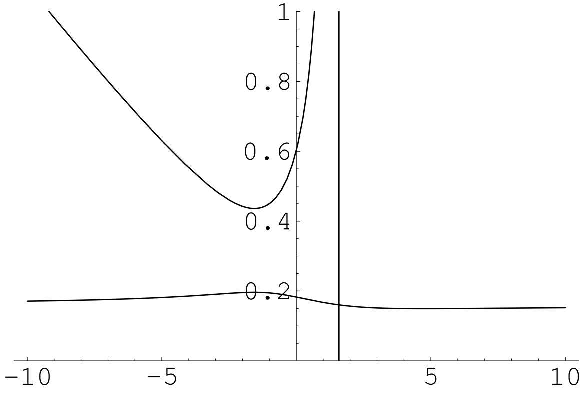


(a) 100 Ω

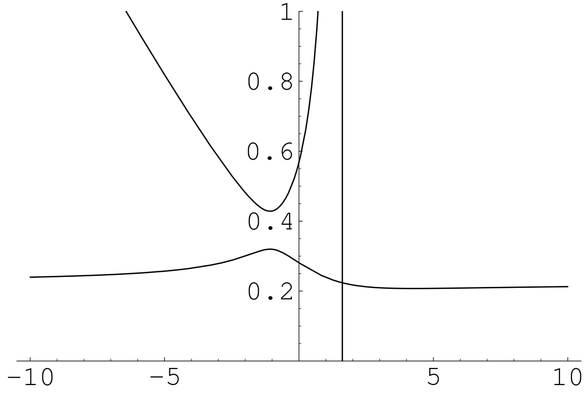
Hopf bifurcation set in k_1-k_2 plane



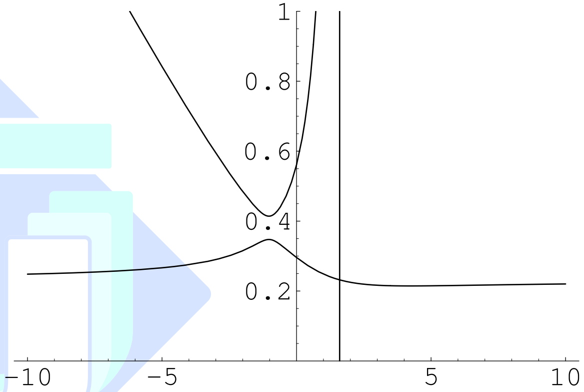
Bifurcation of Hopf bifurcation set



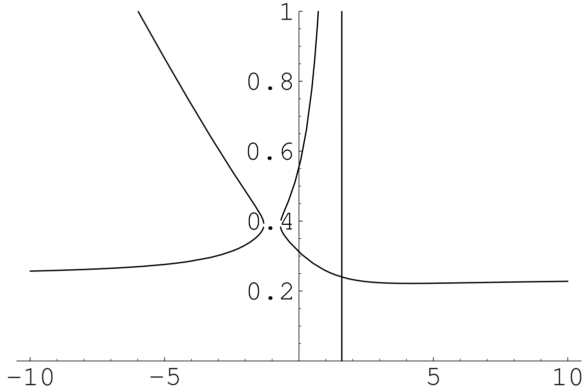
(a) $\alpha = 0.1$.



(b) $\alpha = 0.14$.

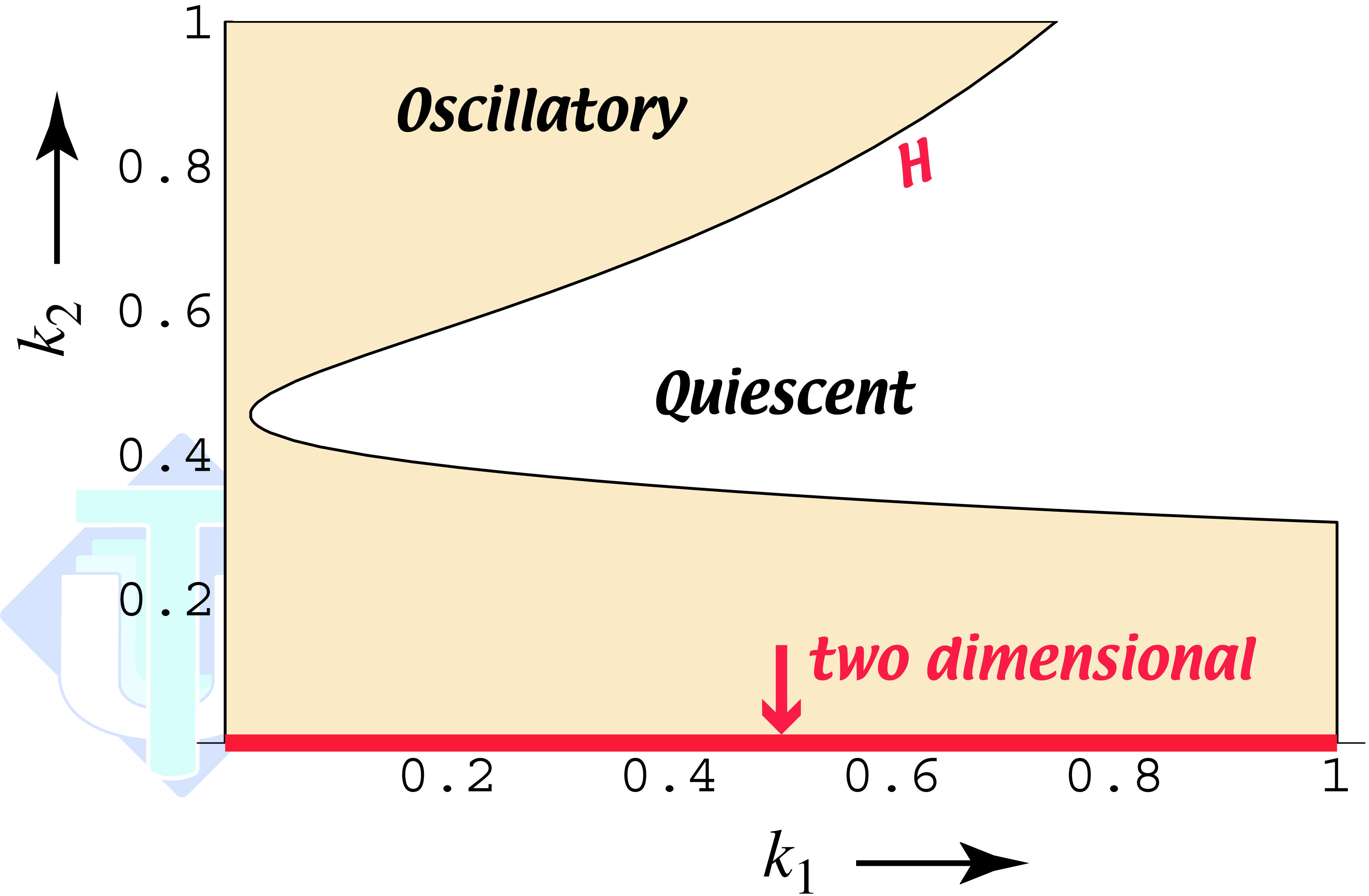


(c) $\alpha = 0.145$.



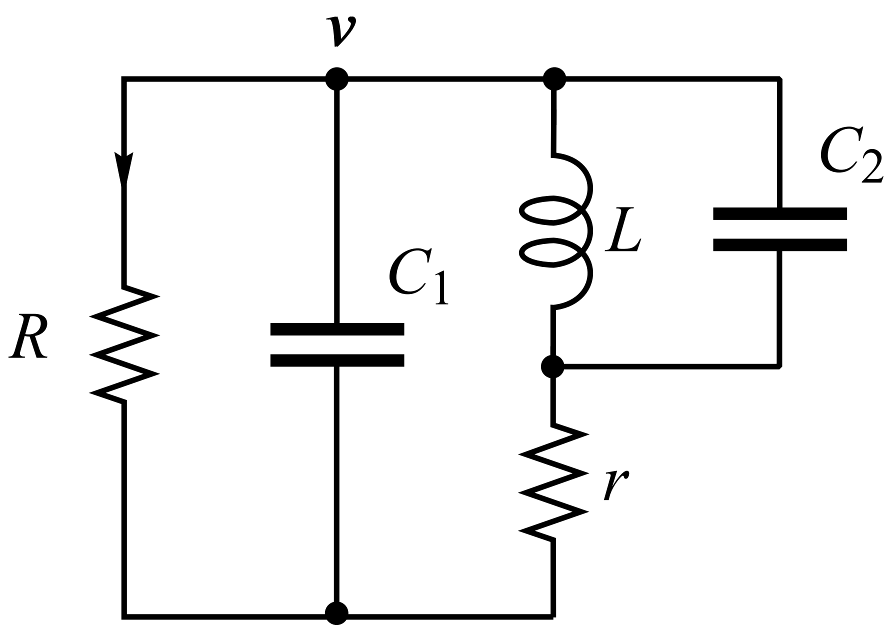
(d) $\alpha = 0.15$.

$\alpha = 0.18$



1
0.8
0.6
0.4
0.2
Oscillatory
H
Quiescent
two dimensional
0.2
0.4
0.6
0.8
1
k_2
k_1

extended BVP circuit 1/5



$$Z = R \parallel \frac{1}{j\omega C_1} \parallel \left(j\omega L + \left(\frac{1}{j\omega C_2} \parallel r \right) \right)$$

extended BVP circuit 2/5

$$Z = \frac{R(r - \omega^2 C_2 r L + j\omega L)}{R(1 + j\omega C_2 r) + (1 + j\omega C_1 R)(r - \omega^2 C_2 r L + j\omega L)}$$

The denominator $\equiv 0 + j0$

$$R + r - \omega^2 (C_2 r - C_1 R) L = 0$$

$$(C_1 + C_2) r R + L - \omega^2 C_1 C_2 r R L = 0$$

extended BVP circuit 3/5

Frequency:

$$\omega = \sqrt{\frac{L + (C_1 + C_2)rR}{C_1 C_2 rRL}}$$

Bifurcation set:

$$(L + (C_1 + C_2)rR)(C_2 r + C_1 R)L - (R + r)C_1 C_2 rRL = 0$$

⇒ gives Hopf bifurcation set in r-R plane.

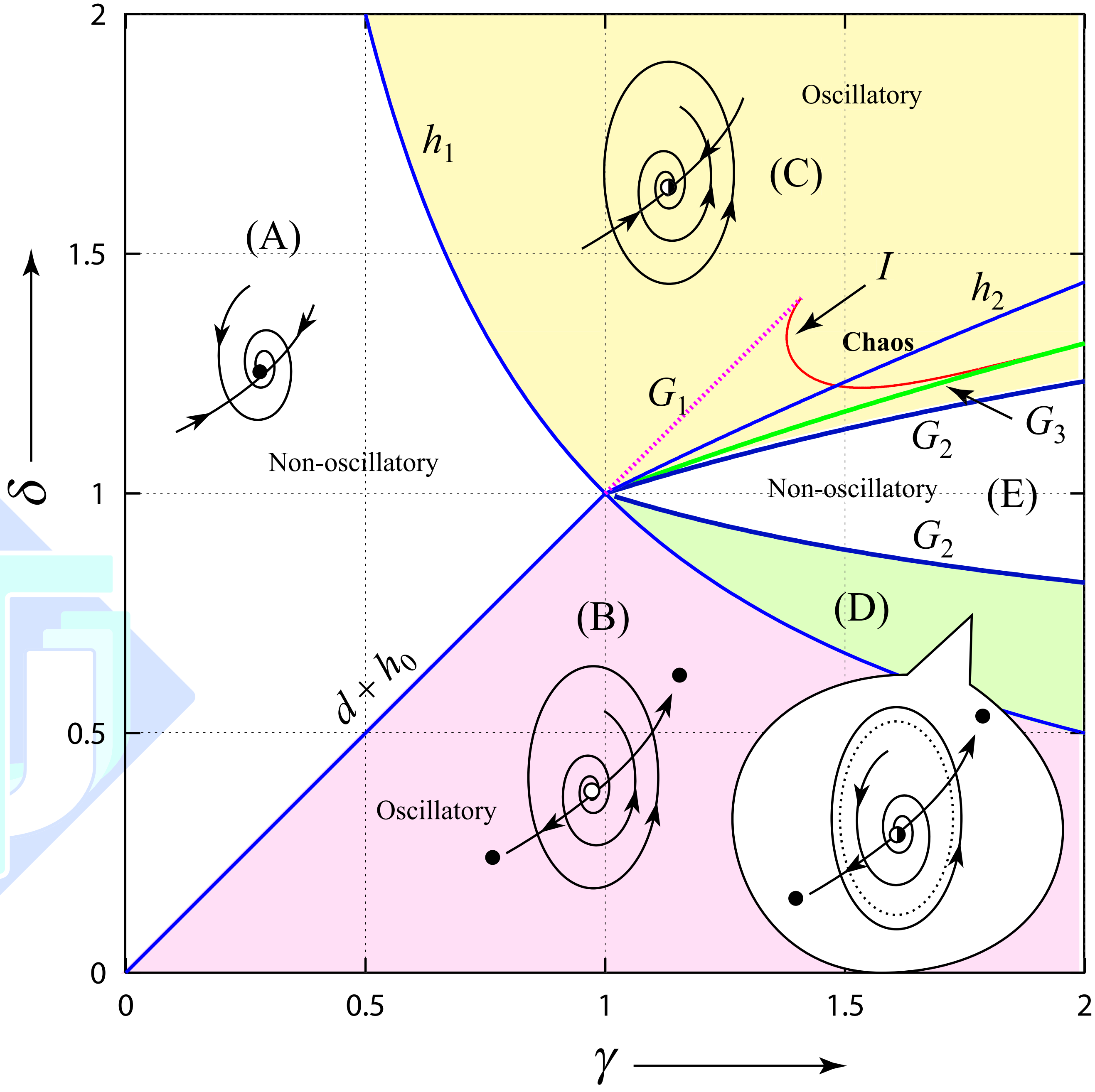
extended BVP circuit 4/5

$$r = \frac{-C_2L - C_1^2R^2 \pm \sqrt{-4C_1C_2^2LR^2 + (C_2L + C_1^2R^2)^2}}{2C^2R}$$

In case that $C_1 = C_2 = C$:

$$rR = -\frac{L}{C}, \quad r = -R$$

extended BVP circuit 5/5



REMARKS

Oscillator design:

- ✎ **Hopf bifurcation analysis, Barkhausen criterion, virtual source method**
- ✎ **Hopf bifurcation set: oscillation condition curve itself**
- ✎ **Phasor method is easer to analyze the system**